ON TRANSGRESSION IN ASSOCIATED BUNDLES

ZHAOHU NIE

Abstract. We formulate and prove a formula for transgressing characteristic forms in general associated bundles following a method of Chern [Che91]. As applications, we derive D. Johnson's explicit formula in [Joh07] for such general transgression and Chern's first transgression formula in [Che45] for the Euler class.

1. Introduction

Let G be a Lie group with Lie algebra $\mathfrak{g}$, M a manifold, and $\pi : E \to M$ a principal G-bundle over M. A connection is given by a $\mathfrak{g}$-valued 1-form ω on E satisfying certain conditions. Its curvature form is a $\mathfrak{g}$-valued 2-form on E defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{1.1}$$

For $P \in \mathcal{I}(\mathfrak{g})$ an ad_G-invariant polynomial on $\mathfrak{g}$, the form $P(\Omega)$, *a priori* defined on E, is horizontal and invariant and so naturally defines a form on M. $P(\Omega)$ is closed and its cohomology class is independent of the choice of the connection ω. As such, it is called a characteristic form of E.

Chern-Simons [CS74] transgressed $P(\Omega)$ in the principal bundle E. That is, they showed that $P(\Omega)$ is a coboundary in E by canonically constructing a form $TP(\omega)$ (T for transgression), depending on the connection ω, such that $dTP(\omega) = P(\Omega)$. The Chern-Simons forms $TP(\omega)$ define important secondary invariants and appear naturally in questions involving manifolds with boundaries.

It is also important to be able to transgress the characteristic form $P(\Omega)$ in "smaller" bunldes, that is, associated bundles. Let $H < G$ be a subgroup with Lie algebra $\mathfrak{h}$. We assume that the homogenous space G/H is *reductive*. Consider the associated bundle $B = E \times_G (G/H)$, which fits in the following commutative diagram

$$\begin{tikzcd} E \arrow[rr, "\pi_1"] \arrow[dr, "\pi"'] & & B \arrow[dl, "\pi_2"] \\ & M. & \end{tikzcd}$$

Here $\pi_1 : E \to B$ is a principal bundle with structure group H. The question is then to find a canonical form $TP(\omega)$ on the associated bundle B, which transgresses $P(\Omega)$ (regarded as on B), at least in many important cases. (See the precise statement in (2.12).)

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Chern [Che91] solved this question for the important special case of Chern classes c_s with the relevant groups being $GL(s-1;\mathbb{C}) \subset GL(q;\mathbb{C})$ for $1 \le s \le q$, using a natural deformation trick with its root in [CS74]. On the other hand, in the general situation, D. Johnson [Joh07] explicitly constructed $TP(\omega)$ by (rather complicated) recurrence.

In this paper, we first formulate and prove a transgression formula in general associated bundles in Section 2. The setup we consider follows that of D. Johnson [Joh07], but the idea of construction comes from Chern [Che91]. Then we derive D. Johnson's explicit transgression formula in [Joh07] rather easily in Section 3. In Section 4, we show that the very first transgression formula which started the whole business, that is, Chern's formula in [Che45] for transgressing the Euler class in the unit tangent sphere bundle, can be obtained using our general method. This shows compatibility of different approaches to transgression throughout history.

2. General transgression formula

With notation as above and following D. Johnson [Joh07], we choose and fix an ad_H-invariant decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}. \tag{2.1}$$

We denote the corresponding projections by $p_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $p_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$. Then one has the following decomposition of ω

$$\omega = \psi + \phi, \tag{2.2}$$

$$\psi = p_{\mathfrak{h}} \circ \omega \in \Lambda^1(E, \mathfrak{h}),\ \ \phi = p_{\mathfrak{p}} \circ \omega \in \Lambda^1(E, \mathfrak{p}). \tag{2.3}$$

ψ is easily seen to be a connection form on $\pi_1 : E \to B$ with curvature form $\Psi = d\psi + \frac{1}{2}[\psi, \psi]$. By (1.1), we calculate the curvature form of ω to be

$$\begin{aligned} \Omega &= d(\psi + \phi) + \frac{1}{2}[\psi + \phi, \psi + \phi] \\ &= d\psi + \frac{1}{2}[\psi, \psi] + d\phi + [\psi, \phi] + \frac{1}{2}[\phi, \phi] \\ &= \Psi + d_{\mathcal{H}}\phi + \frac{1}{2}[\phi, \phi], \end{aligned} \tag{2.4}$$

where

$$d_{\mathcal{H}}\phi = d\phi + [\psi, \phi] \tag{2.5}$$

is the ψ-covariant derivative of ϕ. One then has from (2.4)

$$d_{\mathcal{H}}\phi = \Omega - \Psi - \frac{1}{2}[\phi, \phi]. \tag{2.6}$$

Following [Che91], consider the following family of differential forms on E

$$\omega(t) = \psi + t\phi,\ \ 0 \le t \le 1. \tag{2.7}$$

Following (1.1), define

$$\Omega(t) = d\omega(t) + \frac{1}{2}[\omega(t), \omega(t)]. \tag{2.8}$$

Similar to (2.4), one calculates

$$\Omega(t) = \Psi + t\, d_{\mathcal{H}}\phi + \frac{1}{2}t^2[\phi,\phi] \tag{2.9}$$

$$= (1-t)\Psi - \frac{1}{2}t(1-t)[\phi,\phi] + t\Omega. \tag{2.10}$$

where the last equality uses (2.6).

We polarize an invariant polynomial $P \in \mathcal{I}^k(\mathfrak{g})$ of degree k to a symmetric, multi-linear function $P : \underbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}_{k} \to \mathbb{R}$. Chern [Che91] proved the following theorem in the special case of Chern classes.

Theorem 1 ([Che91]). *Notation as above. There is a differential form on the associated bundle B defined by*

$$TP(\omega) = k\int_0^1 P(\phi,\Omega(t),\ldots,\Omega(t))\,dt \tag{2.11}$$

such that

$$dTP(\omega) = P(\Omega) - P(\Psi), \tag{2.12}$$

For the reader's convenience, we give a brief proof of this theorem in our more general situation.

Proof. We will show that $TP(\omega)$ in (2.11) defines naturally a differential form on B by showing that it is invariant and horizontal for the principal bundle $\pi_1 : E \to B$ with connection ψ. It is easy to see that for the right multiplication R_h by $h \in H$, one has

$$R_h^*\psi = \mathrm{ad}_{h^{-1}}\,\psi,\ \ R_h^*\phi = \mathrm{ad}_{h^{-1}}\,\phi,$$

since ω is a connection form and the decomposition in (2.1) is ad_H-invariant. Therefore by (2.7) and (2.8), one has $R_h^*\Omega(t) = \mathrm{ad}_{h^{-1}}\,\Omega(t)$. The invariance of P under ad_G and hence under ad_H then shows that $TP(\omega)$ is invariant. It is also easy to see that ϕ is horizontal from its definition in (2.3). The horizontality of curvature forms Ψ and Ω then implies the horizontality of $\Omega(t)$ from (2.10) and that of $TP(\omega)$.

In (2.10), $\Omega(1) = \Omega$ and $\Omega(0) = \Psi$. Therefore to prove (2.12), we only need to show that

$$\frac{d}{dt}P(\Omega(t)) = kdP(\phi,\Omega(t),\ldots,\Omega(t)).$$

One computes

$$\begin{aligned}\frac{d}{dt}P(\Omega(t)) &= kP(\frac{d}{dt}\Omega(t),\Omega(t),\ldots,\Omega(t))\\ &= kP(d_{\mathcal{H}}\phi + t[\phi,\phi],\Omega(t),\ldots,\Omega(t))\end{aligned}$$

by (2.9). Also

$$\begin{aligned}&dP(\phi,\Omega(t),\ldots,\Omega(t))\\ =&P(d_{\mathcal{H}}\phi,\Omega(t),\ldots,\Omega(t)) - (k-1)P(\phi,d_{\mathcal{H}}\Omega(t),\Omega(t),\ldots,\Omega(t))\\ =&P(d_{\mathcal{H}}\phi,\Omega(t),\ldots,\Omega(t)) - (k-1)P(\phi,t[\Omega(t),\phi],\Omega(t),\ldots,\Omega(t))\\ =&P(d_{\mathcal{H}}\phi,\Omega(t),\ldots,\Omega(t)) + P(t[\phi,\phi],\Omega(t),\ldots,\Omega(t))\\ =&P(d_{\mathcal{H}}\phi + t[\phi,\phi],\Omega(t),\ldots,\Omega(t)),\end{aligned}$$

where the second equality uses

$$\begin{aligned}&d_{\mathcal{H}}\Omega(t) = d\Omega(t) + [\psi, \Omega(t)]\\ =&[d\omega(t), \omega(t)] - [\Omega(t), \psi] = [\Omega(t), \omega(t)] - [\Omega(t), \psi]\\ =&t[\Omega(t), \phi]\end{aligned}$$

by differentiating (2.8) and using (2.7), and the third uses the adjoint-invariance of P, which implies

$$P([\phi, \phi], \Omega(t), \ldots, \Omega(t)) + (k-1)P(\phi, [\Omega(t), \phi], \Omega(t), \ldots, \Omega(t)) = 0.$$

□

3. D. Johnson's explicit formula

As in [CS74, (3.5)], one can evaluate the integral formula in (2.11) using (2.10), which then gives the following explicit formula of D. Johnson [Joh07], who proved it using recurrence.

Theorem 2 ([Joh07])**.** *Notation as above. The explicit formula for the transgression form on an associated bundle is*

$$TP(\omega) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} A_{ij} P(\phi, [\phi, \phi]^i, \Psi^j, \Omega^{k-i-j-1}), \tag{3.1}$$

where $A_{ij} = (-1)^i \frac{k!(k-j-1)!(i+j)!}{2^i i! j! (k-i-j-1)!(k+i)!}$.

Proof using Theorem 1. Plugging (2.10) into (2.11), applying the multinomial theorem for the $(k-1)$ arguments of $\Omega(t)$ (since P is symmetric and the differential forms involved are of degree two), and applying some basic knowledge about the beta functions, one gets

$$\begin{aligned}TP(\omega) &= k\int_0^1 P\Big(\phi, \underbrace{-\frac{1}{2}t(1-t)[\phi,\phi] + (1-t)\Psi + t\Omega, \ldots}_{k-1}\Big)\, dt\\ =&k\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \frac{(k-1)!}{i!j!(k-i-j-1)!} \int_0^1 \Big(-\frac{1}{2}t(1-t)\Big)^i (1-t)^j t^{k-i-j-1}\, dt\\ &\qquad P(\phi, [\phi,\phi]^i, \Psi^j, \Omega^{k-i-j-1})\\ =&k\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \frac{(k-1)!}{i!j!(k-i-j-1)!} \Big(-\frac{1}{2}\Big)^i \int_0^1 t^{k-j-1}(1-t)^{i+j}\, dt\\ &\qquad P(\phi, [\phi,\phi]^i, \Psi^j, \Omega^{k-i-j-1})\\ =&\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \frac{k!}{i!j!(k-i-j-1)!} \Big(-\frac{1}{2}\Big)^i \frac{(k-j-1)!(i+j)!}{(k+i)!}\\ &\qquad P(\phi, [\phi,\phi]^i, \Psi^j, \Omega^{k-i-j-1}).\end{aligned}$$

This is (3.1). □

4. Chern's first transgression formula

As another example, we work out Chern's first transgression formula in [Che45] for the Euler form using Theorem 1. ([Che45] reformulates and simplifies the truly "first" formula in [Che44].)

Assume M has even dimension $n = 2k$. For the Euler form, the relevant Lie groups are $SO(n-1) \subset SO(n)$, and the associated bundle $B \to M$ is the unit tangent sphere bundle $STM \to M$. We use the convention that i ranges from 1 to n, and α, β range from 1 to $n-1$. The invariant polynomial of degree k on $\mathfrak{so}(n)$ for the Euler form is the Pfaffian up to a scale, defined by

$$\widetilde{Pf}(A) = \frac{1}{(2\pi)^k} \frac{(-1)^k}{2^k k!} \sum_i \epsilon(i) A_{i_1 i_2} \dots A_{i_{n-1} i_n} \tag{4.1}$$

for $A \in \mathfrak{so}(n)$, where the summation ranges over permutations of $\{1, \dots, n\}$. The coefficients are present to make the corresponding characteristic class $\widetilde{Pf}(\Omega)$ integral.

Theorem 3 ([Che45])**.** *There is a differential form on the unit tangent sphere bundle STM defined by*

(4.2)

$$T\widetilde{Pf}(\omega) = \frac{1}{(2\pi)^k} \sum_{j=0}^{k-1} \frac{(-1)^{j+1}}{2^j j!(2k-2j-1)!!} \sum_\alpha \epsilon(\alpha) \Omega_{\alpha_1 \alpha_2} \dots \Omega_{\alpha_{2j-1}\alpha_{2j}} \omega_{\alpha_{2j+1} n} \dots \omega_{\alpha_{n-1} n}$$

such that

$$dT\widetilde{Pf}(\omega) = \widetilde{Pf}(\Omega). \tag{4.3}$$

Remark 4.4. [Che45, (9),(11)] solved $d\Pi = -\widetilde{Pf}(\Omega)$, negative of the Euler form. This accounts for the exponent of -1 being $j+1$ in (4.2) instead of the usual j.

Proof using Theorem 1. Write the connection form ω with values in skew-symmetric matrices as

$$\omega = \begin{pmatrix} \omega_{\alpha\beta} & \omega_{\alpha n} \\ -\omega_{\beta n} & 0 \end{pmatrix}.$$

By (2.2) and with the obvious decomposition, we have

$$\psi = \begin{pmatrix} \omega_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \ \phi = \begin{pmatrix} 0 & \omega_{\alpha n} \\ -\omega_{\beta n} & 0 \end{pmatrix}. \tag{4.5}$$

We compute

$$\frac{1}{2}[\phi, \phi] = \begin{pmatrix} \omega_{\alpha n}\omega_{\beta n} & 0 \\ 0 & 0 \end{pmatrix}, \ \Omega = \begin{pmatrix} \Omega_{\alpha\beta} & \Omega_{\alpha n} \\ \Omega_{n\beta} & 0 \end{pmatrix}. \tag{4.6}$$

When applying formula (2.11) for $\widetilde{Pf}$ in (4.1), one has the factor $\phi_{i_1 i_2}$. Therefore in view of (4.5), one of the i_1, i_2 must be n for $\phi_{i_1 i_2}$ to be nonzero. As a result, one only cares about the (α, β)-elements of $\Omega(t)$ in (2.11) in our situation. One important thing to note is that from (2.5) and (4.5), the ψ-covariant derivative of ϕ

$$d_{\mathcal{H}}\phi = d\phi + [\psi, \phi] = \begin{pmatrix} 0 & d\omega_{\alpha n} \\ -d\omega_{\beta n} & 0 \end{pmatrix}$$

has trivial (α,β)-elements. Hence we say $d_{\mathcal{H}}\phi \equiv 0$ with " $\equiv$ " meaning "having the same (α,β)-elements". Then (2.4) gives $\Psi \equiv \Omega - \frac{1}{2}[\phi,\phi]$ and (2.9) gives

$$\Omega(t) \equiv \Omega - \frac{1}{2}(1-t^2)[\phi,\phi]. \tag{4.7}$$

We now compute formula (2.11) using (4.7) and the idea for the proof of Theorem 2, rather than computing directly (3.1) which may be harder. That is, we will apply the binomial theorem and some basic integral formula, which in this case is

$$\int_0^1 \left(1-t^2\right)^{k-j-1}\, dt = \frac{(2k-2j-2)!!}{(2k-2j-1)!!}$$

by trigonometric substitution and induction using integration by parts. We proceed as follows:

$$T\widetilde{Pf}(\omega) = k\int_0^1 \widetilde{Pf}\Big(\phi, \underbrace{\Omega - \frac{1}{2}(1-t^2)[\phi,\phi], \dots}_{k-1}\Big)\, dt$$

$$=k\sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-j-1)!}\int_0^1 \left(-(1-t^2)\right)^{k-j-1} dt\, \widetilde{Pf}\left(\phi, \Omega^j, \left(\frac{1}{2}[\phi,\phi]\right)^{k-j-1}\right)$$

$$=\sum_{j=0}^{k-1} \frac{k!}{j!(k-j-1)!}(-1)^{k-j-1}\frac{(2k-2j-2)!!}{(2k-2j-1)!!}\,\widetilde{Pf}\left(\phi, \Omega^j, \left(\frac{1}{2}[\phi,\phi]\right)^{k-j-1}\right)$$

$$=\sum_{j=0}^{k-1} \frac{k!}{j!(k-j-1)!}(-1)^{k-j-1}\frac{2^{k-j-1}(k-j-1)!}{(2k-2j-1)!!}\frac{1}{(2\pi)^k}\frac{(-1)^k}{2^k k!}$$

$$\sum_i \epsilon(i)\phi_{i_1 i_2}\Omega_{i_3 i_4}\dots\Omega_{i_{2j+1}i_{2j+2}}\omega_{i_{2j+3}n}\omega_{i_{2j+4}n}\dots\omega_{i_{n-1}n}\omega_{i_n n}$$

$$=\frac{1}{(2\pi)^k}\sum_{j=0}^{k-1}\frac{(-1)^{j+1}}{2^{j+1}j!(2k-2j-1)!!}\mathbf{2}\sum_\alpha \epsilon(\alpha)\omega_{\alpha_1 n}\Omega_{\alpha_2\alpha_3}\dots\Omega_{\alpha_{2j}\alpha_{2j+1}}\omega_{\alpha_{2j+2}n}\dots\omega_{\alpha_{n-1}n},$$

where the second last equality follows from (4.1) and (4.6), and the last equality follows after some cancellation of coefficients and from (4.5). Here the **2** appears because n can appear as i_1 or i_2 in $\phi_{i_1 i_2}$ and the summation ranges over permutations of $\{1,\dots,n-1\}$. The last expression is exactly (4.2), after canceling the **2** and moving the $\omega_{\alpha_1 n}$ after the Ω's.

Now $\Psi = d\psi + \frac{1}{2}[\psi,\psi]$ clearly has the last row and column zero from (4.5). One then has $\widetilde{Pf}(\Psi) = 0$ for our $\widetilde{Pf}$ in (4.1). Therefore (2.12) implies (4.3). $\square$

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E-mail address: `znie@psu.edu`

Department of Mathematics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA